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Application of Numerical Method to Wave Equation

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Abstract—*In this paper, the finite – difference approach, the continuous problem domain is "discretized", so that the dependent variables exists only at discrete points. Derivatives are approximated by differences, resulting in an algebraic representation of the partial differential equation (PDE). Hence a problem involving calculus transforms into an algebraic problem.*
Multiple considerations determine whether the solution so obtained will be a good approximation to the exact solution of the original PDE. Among these considerations are truncation error, round off error and consistency, all of which is discussed in the current paper. There are two types of numerical method for solving mathematical equations. The first type approximates the unknown function in the equation by a simple function often a polynomial or piecewise polynomial function, chosen to closely follow the original equation. The second type of numerical method approximates the equation of interest, usually by approximating the derivatives of integrals in the equation. Such numerical procedures are often called finite difference methods. Most initial value problems for ordinary differential equations and partial differential equation are solved using this method Numerical method for solving differential and integral equation. involves both approximation theory and the solution of large and non linear system of equations..

1. INTRODUCTION

In this paper, basic concepts and techniques needed in the formulation of finite – difference and finite – volume representations are developed. In the finite – difference approach, the continuous problem domain is "discretized", so that the dependent variables are considered to exist only at discrete points. Derivatives are approximated by differences, resulting in an algebraic problem.

The nature of the resulting algebraic system depends on the character of the problem posed by the original PDE. Equilibrium problems usually result in a system of algebraic equations that must be solved simultaneously throughout the problem domain in conjunction with specified boundary values. Marching problems result in algebraic equations that usually can be solved one at a time (although it is often convenient to solve them several at a time). Several considerations determine whether the solution so obtained will be a good approximation to the exact solution of the original PDE. Among these considerations are truncation error, round – off and consistency, all of which will be discussed in the present work

Finite Differences

One of the first steps to be taken in establishing a finite – difference procedure for solving a PDE is to replace the continuous problem domain by a finite difference mesh or grid. As an example, suppose that we wish to solve a PDE for which u(x, y) is the dependent variable in the square domain $0 \leq x \leq 1$, $0 \leq y \leq 1$. We establish a grid on the domain by replacing u(x, y) by u (iΔx, jΔy). Points can be located according to values of i and j, so difference equations are usually written in terms of the general point

2. DIFFERENCE REPRESENTATIONS OF PARTIAL DIFFERENTIAL EQUATIONS

Truncation Error

As a starting point in our study of T.E., let us consider the heat equation

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} \quad (3.1)$$

Using a forward-difference representation for the time derivative $(t = n\Delta t)$ and a central-difference representation for the second derivative, we can approximate the heat equation by

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{\alpha}{(\Delta x)^2}\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right) \quad (3.2a)$$

However, we noted that T.E.s were associated with the forward and central-difference representations used in Eq. (3.2). If we rearrange Eq. (3.1) to put zero on the right-hand side and include the T.E.s associated with the difference representation of the derivatives,

Wave Equation

The one-dimensional (1-D) wave equation is a second-order hyperbolic PDE given by

$$\frac{\partial^2 u}{\partial t^2} = c\frac{\partial^2 u}{\partial x^2} \text{ (3.3)}$$

This equation governs the propagation of sound waves traveling at a wave speed c in a uniform medium. A first-order equation that has properties similar to those of Eq.(2.3) is given by

$$\frac{\partial u}{\partial t} + c\,\frac{\partial u}{\partial x} = 0 \; c > 0 \text{ (3.4)}$$

Note that Eq. (3.3) can be obtained from Eq. (3.4).We will use Eq.(3.3) as our model equation and refer to it as the first-order 1-D wave equation, or simply the "wave equation". This linear hyperbolic equation describes a wave propagating in the x direction with a velocity c, and it can be used to "model" in a rudimentary fashion the nonlinear equations governing inviscid flow. Although we will refer to Eq. (3.4) as the wave equation, the reader is cautioned to be aware of the fact that Eq. (3.3) is the classical wave equation. More appropriately, Eq. (3.4) is often called the 1- D linear convection equation.

The exact solution of the wave equation Eq. (3.4) for the pure initial value problem with initial data

$$u(x,0) = F(x) \; -\infty < x \text{ (3.5)}$$

Is given by

$$u(x,t) = F(x-ct) \text{ (3.6)}$$

Let us now examine some schemes that could be used to solve the wave equation.

Euler Explicit Methods:

The following simple explicit one-step methods,

$$\frac{u_j^{n+1}-u_j^n}{\Delta t} + c\frac{u_{j+1}^n-u_j^n}{\Delta x} = 0 \; c > 0 \text{(3.7)}$$

$$\frac{u_j^{n+1}-u_j^n}{\Delta t} + c\frac{u_{j+1}^n-u_j^n}{2\Delta x} = 0 \text{ (3.8)}$$

Have truncation errors (T.E.s) of O [Δt, Δx] and O [Δt,$(\Delta x)^2$], respectively. We refer to these schemes as being first-order accurate, since the lowest-order term in the T.E. is first order, i.e., Δt and Δx .. These schemes are explicit, since only one unknown u_j^{n+1} appears in each equation.

Unfortunately, when the von Neumann stability analysis is applied to these schemes, we find that they are unconditionally unstable. These simple schemes, therefore, prove to be worthless in solving the wave equation. Let us now proceed to look at methods that have more utility.

Upstream (First-Order Upwind or Windward) Differencing Method

The simple Euler method, can be made stable by replacing the forward space difference by a backward space difference, provided that the wave speed c is positive. If the wave speed is negative, a forward difference must be used to assure stability. For a positive wave speed, the following algorithm results:

$$\frac{u_j^{n+1}-u_j^n}{\Delta t} + c\frac{u_j^n-u_{j-1}^n}{\Delta x} = 0 \; c > 0$$

This is a first-order accurate method with T.E. of O[Δt, Δx].The von Neumann stability analysis shows that this method is stable, provided that

0 ≤ v ≤ 1 (3.9)

Where v $= \frac{c\Delta t}{\Delta x}$.

Let us substitute Taylor-series expansions into Eq. for u_j^{n+1} and u_{j-1}^n.

The following equation result:

$$\frac{1}{\Delta t}\left\{\left[u_j^n + \Delta t u_t + \frac{(\Delta t)^2}{2}u_{tt} + \frac{(\Delta t)^3}{6}u_{ttt} + \cdots\right] - u_j^n\right\} + \frac{c}{\Delta x}\left\{u_j^n - [u_j^n - \Delta x u_x + \frac{(\Delta x)^2}{2}u_{xx} - \frac{(\Delta x)^3}{6}u_{xxx} + \cdots]\right\} = 0 \text{ (3.10)}$$

Equation (2.10) simplifies to

$$u_t + cu_x = -\frac{\Delta t}{2}u_{tt} + \frac{c\Delta x}{2}u_{xx} - \frac{(\Delta t)^2}{6}u_{ttt} - c\frac{(\Delta x)^2}{6}u_{xxx} + \cdots \text{ (3.11)}$$

Note that the left-hand side of this equation corresponds to the wave equation and the right-hand side is the T.E., which is generally not zero. The significance of terms in the T.E. can be more easily interpreted

$$u_{tt} + cu_{xt} = -\frac{\Delta t}{2}u_{\square tt} + \frac{c\Delta x}{2}u_{xxt} - \frac{(\Delta t)^2}{6}u_{tttt} - c\frac{(\Delta x)^2}{6}u_{xxxt} + \cdots \text{ (3.12)}$$

And take the partial derivative of Eq. (3.10) with respect to x and multiply by –c:

$$-cu_{xx} - c^2u_{xx} = \frac{c\Delta t}{2}u_{ttx} - \frac{c^2\Delta x}{2}u_{xxx} + \frac{c(\Delta t)^2}{6}u_{tttx} + \frac{c^2(\Delta x)^2}{6}u_{xxxx} + \ldots$$

(3.13)

Adding Eqs. (3.12) and (3.13) gives

$$u_{tt} = c^2u_{xx} + \Delta t\left(\frac{-u_{ttt}}{2} + \frac{c}{2}u_{ttx} + O[\Delta t]\right) + \Delta x\left(\frac{c}{2}u_{xxt} - \frac{c^2}{2}u_{xxx} + O[\Delta x]\right)$$

(3.14)

In a similar manner, we can obtain the following expressions for u_{ttt}, u_{ttx} *and* u_{xxt}

$$u_{ttt} = -c^3 u_{xxx} + O[\Delta t, \Delta x]$$
$$u_{ttx} = c^2 u_{xxx} + O[\Delta t, \Delta x]$$
$$u_{xxt} = -c u_{xxx} + O[\Delta t, \Delta x]$$

(3.15)

Combining Eqs. (3.11),(3.14) and (3.15) gives

$$u_t + cu_x = \frac{c\Delta x}{2}(1-v)u_{xx} - \frac{c(\Delta x)^2}{6}(2v^2 - 3v + 1)u_{xxx} + O[(\Delta x)^3, (\Delta x)^2\Delta t, \Delta x(\Delta t)^2, (\Delta t)^3]$$

(3.16)

An equation, such as Eq. (3.16), is called a modified equation. It can be thought of as the PDE that is actually solved (if sufficient boundary conditions were available) when a finite-difference method

The right-hand side of the modified equation [Eq. (3.16)] is the T.E., since it represents the difference between the original PDE and the finite-difference approximation to it. Consequently, the lowest order term on the right-hand side of the modified equation gives the order of the method. In the present case, the method is first- order accurate, since the lowest order term is $[\Delta t, \Delta x]$. If v=1, the right- hand side of the modified equation becomes zero, and the wave equation is solved exactly. In this case, the upstream differencing scheme reduces to

$$u_j^{n+1} = u_{j-1}^n$$

Which is equivalent to solving the wave equation exactly using the method of characteristics. Finite-difference algorithms that exhibit this behavior are said to satisfy the shift condition.

The lowest order term of the T.E. in the present case contains the partial derivative u_{xx} , which makes this term similar to the viscous term in 1-D fluid flow equations. For example, the viscous term in the 1-D Nevier-stokes equation may be written as

$$\frac{\partial}{\partial x}(\tau_{xx}) = \frac{4}{3}\mu u_{xx} \qquad (3.17)$$

If a constant coefficient of viscosity is assumed. Thus, when v ≠ 1, the upstream differencing scheme introduces an artificial viscosity into the solution. This is often called implicit artificial viscosity, as opposed to explicit artificial viscosity, which is purposely added to a difference scheme. Artificial viscosity tends to reduce all gradients in the solution whether physically correct or numerically induced. This effect, which is the direct result of even derivative terms in the T.E., is called dissipation.

Another quasi- physical effect of numerical schemes is called dispersion. This is the direct result of the odd derivative terms that appear in the T.E. As a result of dispersion, phase relations between various waves are distorted. The combined effect of dissipation and dispersion is sometimes referred to as diffusion.

3. CONCLUSION

We obtained many new solutions of the wave equation using discretization. Methods and extended these method with the help of a suitable transformation. The computer symbolic system such as maple and mathematical allow us to perform complicated and tedious calculations. The solutions have different physical structures and depend on the boundary conditions. It is concluded that the “Upstream method″ are very efficient in finding solution for the wave equation.

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